Stochastic control of SDEs driven by sub-diffusions

Zhen-Qing Chen

University of Washington

Joint work (ongoing) with Shuaiqi Zhang

17th Workshop on Markov Processes and Related Topics

BNU at Zhuhai , November 27, 2022

Sub-diffusion

Anomalous subdiffusions that describe particles move slower than Brownian motion (or the original underlying spatial motion), for example, due to particle sticking and trapping.

A prototype of subdiffusion can be modeled by Brownian motion time-changed by an inverse stable subordinator.
Continuous time random walk model:

$$X_n = \sum_{k=1}^{n} \xi_k, \qquad T_n = \sum_{j=1}^{n} \eta_j,$$

where ξ_k is the kth displacement and η_j is the jth waiting or holding time. Let $N_t = \max\{n : T_n \le t\}$. Then $Y_t = X_{N_t}$ is the CTRW.

Sub-diffusion

Anomalous sub-diffusion is the scaling limit of CTRW when the inter-arrival times $\{\eta_j\}$ have power law tail distribution and the displacements $\{\xi_k\}$ have zero mean and finite variance.

Let B is Brownian motion in $\mathbb{R}^d$ and S an β-stable subordinator. Define

$$E_t = \sup\{r > 0 : S_r \le t\} = \inf\{r > 0 : S_r > t\}.$$

Then B_{E_t} provides a model for anomalous sub-diffusion, where particles spread slower than Brownian particles.

Time-fractional equation

Let $u(t,x) = \mathbb{E}_x[f(B_{E_t})]$. Then [Baeumer-Meerschaert, 2001] [Meerschaert-Scheffler, 2004]

$$\partial_t^\beta u = \frac{1}{2}\Delta_x u \quad \text{with } u(0,x) = f(x),$$

where

$$\partial_t^\beta g(t) := \frac{d}{dt}\int_0^t (g(t-r) - g(0))\frac{1}{\Gamma(1-\beta)} r^{-\beta} dr$$

is the the classical Caputo fractional derivative ∂_t^β of order β. (A. N. Kochubei).

Fractional time equation also arises in many other circumstances, including heat propagation in material with thermal memory. B_{E_t} is called fractional-kinetics process in some literature.

Time-fractional equation

Let $u(t,x) = \mathbb{E}_x[f(B_{E_t})]$. Then [Baeumer-Meerschaert, 2001] [Meerschaert-Scheffler, 2004]

$$\partial_t^\beta u = \frac{1}{2}\Delta_x u \quad \text{with } u(0,x) = f(x),$$

where

$$\partial_t^\beta g(t) := \frac{d}{dt}\int_0^t (g(t-r) - g(0))\frac{1}{\Gamma(1-\beta)}r^{-\beta}dr$$

is the the classical Caputo fractional derivative ∂_t^β of order β. (A. N. Kochubei).

Fractional time equation also arises in many other circumstances, including heat propagation in material with thermal memory. B_{E_t} is called fractional-kinetics process in some literature.

General time-fractional equation

If S_t is a general subordinator with Laplace exponent

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx),$$

then $u(t, x) = \mathbb{E}_x[f(B_{E_t})]$ satisfies [C. 2017]

$$(\kappa\partial_t + \partial_t^\nu)u = \frac{1}{2}\Delta_x u \quad \text{with } u(0, x) = f(x),$$

where

$$\begin{aligned} \partial_t^\nu g(t) &:= \frac{d}{dt}\int_0^t (g(t-r) - g(0))\nu(r, \infty)dr \\ &= \int_0^t (g(t-r) - g(0))\nu(dr) \quad \text{if } g \text{ is Lipschitz}. \end{aligned}$$

Stock market

Anomalous sub-diffusions can also be used to model bull and bear markets for stocks and commodities.

$$dS_t = S_t(\mu_t dt + \sigma_t dB_{L_t}).$$

Solution:

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_{L_s} + \int_0^t \mu_s ds - \frac{1}{2}\int_0^t \sigma_s^2 dL_s\right).$$

Stochastic control of SDEs

Given initial time s, initial state x_0 and control process u,

$$\begin{cases} dx^u(t) = b(t, x^u(t), u(t))dt + \sigma(t, x^u(t), u(t))dB_{L_{(t-s-a)^+}}, \\ \qquad\qquad \text{for } t \in [s, T], \\ x^u(s) = x_0. \end{cases}$$

Cost function:

$$J(s, x_0, u, a) \quad = \quad \mathbb{E}\left[\int_s^T f\left(t, x^{u,s,x_0,a}(t), u(t)\right) dt + h(x^{u,s,x_0,a}(T))\right],$$

Optimal control:

$$J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}_a[s,T]} J(s, x_0, u, a) =: V(s, x_0, a).$$

(respectively, $J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}'_a[s,T]} J(s, x_0, u, a)$).

Stochastic control of SDEs

Given initial time s, initial state x_0 and control process u,

$$\begin{cases} dx^u(t) = b(t, x^u(t), u(t))dt + \sigma(t, x^u(t), u(t))dB_{L_{(t-s-a)^+}}, \\ \qquad\qquad \text{for } t \in [s, T], \\ x^u(s) = x_0. \end{cases}$$

Cost function:

$$J(s, x_0, u, a) \quad = \quad \mathbb{E}\left[\int_s^T f\,(t, x^{u,s,x_0,a}(t), u(t))\,dt + h(x^{u,s,x_0,a}(T)) \right],$$

Optimal control:

$$J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}_a[s,T]} J(s, x_0, u, a) =: V(s, x_0, a).$$

(respectively, $J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}'_a[s,T]} J(s, x_0, u, a)$).

Stochastic control of SDEs

Given initial time s, initial state x_0 and control process u,

$$\begin{cases} dx^u(t) = b(t, x^u(t), u(t))dt + \sigma(t, x^u(t), u(t))dB_{L_{(t-s-a)^+}}, \\ \qquad\qquad \text{for } t \in [s, T], \\ x^u(s) = x_0. \end{cases}$$

Cost function:

$$J(s, x_0, u, a) \quad = \quad \mathbb{E}\left[\int_s^T f\,(t, x^{u,s,x_0,a}(t), u(t))\, dt + h(x^{u,s,x_0,a}(T)) \right],$$

Optimal control:

$$J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}_a[s,T]} J(s, x_0, u, a) =: V(s, x_0, a).$$

(respectively, $J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}'_a[s,T]} J(s, x_0, u, a)$).

Novelty

- The control problem for sub-diffusions is *not always stochastic*. In a sense, the control problem is a hybrid of deterministic and stochastic.
- When studying the stochastic maximum principle, the adjoint equation is a backward stochastic differential equation driven by B_{L_t}, which is new.
- The sub-diffusion is not a Markov process. To investigate the dynamic programming principles (DPP), the overshoot process needs to be added to make it Markov. This brings new challenging in the study of the regularity of the value function. The Hamilton-Jacobi-Bellman (HJB) equation has two parts: the interior and the boundary parts.

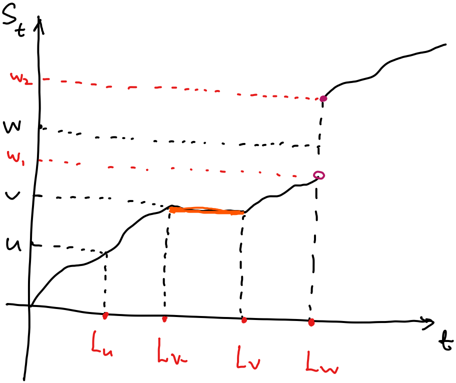
Novelty

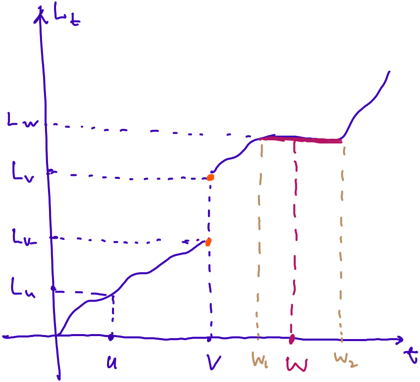
- The control problem for sub-diffusions is *not always stochastic*. In a sense, the control problem is a hybrid of deterministic and stochastic.
- When studying the stochastic maximum principle, the adjoint equation is a backward stochastic differential equation driven by B_{L_t}, which is new.
- The sub-diffusion is not a Markov process. To investigate the dynamic programming principles (DPP), the overshoot process needs to be added to make it Markov. This brings new challenging in the study of the regularity of the value function. The Hamilton-Jacobi-Bellman (HJB) equation has two parts: the interior and the boundary parts.

Novelty

- The control problem for sub-diffusions is *not always stochastic*. In a sense, the control problem is a hybrid of deterministic and stochastic.
- When studying the stochastic maximum principle, the adjoint equation is a backward stochastic differential equation driven by B_{L_t}, which is new.
- The sub-diffusion is not a Markov process. To investigate the dynamic programming principles (DPP), the overshoot process needs to be added to make it Markov. This brings new challenging in the study of the regularity of the value function. The Hamilton-Jacobi-Bellman (HJB) equation has two parts: the interior and the boundary parts.

$$L_t = \inf\{r: S_r > t\}.$$





$S_{L_w} = w_2$ for $w_1 \le w \le w_2$.

When $\kappa > 0$ or S_t has infinite Lévy measure ν, S_t is strictly increasing and so L_t is continuous.

Overshoot process R_t

Recall $L_t = \inf\{r : S_r > t\}$ so $S_{L_t} \geq t$.

Theorem (Zhang-C. 2022+)

Suppose that B is a standard Brownian motion on $\mathbb{R}^d$ starting from 0, S is any subordinator that is independent of B with $S_0 = 0$, and $L_t := \inf\{r > 0 : S_r > t\}$. Then

$$\widetilde{X}_t := (X_t, R_t) := \left(x_0 + B_{L_{(t-R_0)^+}},\ R_0 + S_{L_{(t-R_0)^+}} - t \right), \quad t \geq 0,$$

with $\widetilde{X}_0 = (x_0, R_0)$ is a time-homegenous Markov process taking values in $\mathbb{R}^d \times [0, \infty)$.

$$\begin{aligned} L_{t+s} - L_s &= \inf\{r > 0 : S_r > t + s\} - L_t = \inf\{r > 0 : S_{r+L_t} - S_{L_t} > t + s - S_{L_t}\} \\ &= \inf\left\{r > 0 : S_r \circ \theta_{L_t} > s - (S_{L_t} - t)\right\} = L_{(s-a)^+} \circ \theta_{L_t}, \end{aligned}$$

where $a = S_{L_t} - t$ is the overshoot.

Inverse subordinator

Suppose that $S_t = \kappa t + S_t^0$ with $\kappa > 0$. Its potential measure U has a continuous density function $\vartheta(x) > 0$; that is

$$\mathbb{E} \int_0^\infty f(S_t)dt = \int_{[0,\infty)} f(x)U(dx) = \int_{[0,\infty)} f(x)\vartheta(x)dx.$$

Moreover,

$$\mathbb{P}(S_{L_x} = x) = \kappa\, \vartheta(x) \quad \text{for every } x > 0.$$

Consequently,

$$\frac{d}{dt}\mathbb{E}\left[L_t\right] = \lim_{s\to 0} \frac{U(t+s) - U(t)}{s} = \lim_{s\to 0} \frac{\int_t^{t+s} \vartheta(x)dx}{s} = \vartheta(t)$$

and

$$\lim_{t\to 0} \frac{\mathbb{E}\left[L_t\right]}{t} = \vartheta(0) = 1/\kappa.$$

Definition

(1) For each $0 \le s < T$ and $a \ge 0$, denote by $\mathcal{U}_a[s, T]$ the set of all $\{\mathcal{F}^a_{t-s}\}_{t\in[s,T]}$-progressively measurable processes $\{u(t,\omega); t \in [s, T]\}$ defined on $[s, T] \times \Omega$ with $\mathbb{E}\Big[\sup_{t\in[s,T]} |u(t)|^2 \Big] < \infty$. Here $\{\mathcal{F}^a_t\}_{t\ge 0}$ is the minimum augmented filtration generated by $\widetilde{X} = (X, R)$ with $R_0 = a$.

(2) We say a control $u \in \mathcal{U}'_a[s, T]$ if the filtration $\{\mathcal{F}^a_t\}$ in the above is replaced by the minimum augmented filtration $\{\mathcal{F}^{\prime,a}_t\}$ generated by X, the first coordinate process of $\widetilde{X} = (X, R)$. Clearly, $\mathcal{F}^{\prime,a}_t \subset \mathcal{F}^a_t$ for every $t \ge 0$ and so $\mathcal{U}'_a[s, T] \subset \mathcal{U}_a[s, T]$.

Dynamic programing principle

The following is the counter part of Bellman's principle of optimality for DPP for sub-diffusions.

Theorem (Zhang-C. 2022+)

For any $0 \le s \le \bar{s} \le T$, $y \in \mathbb{R}^d$ *and* $a \ge 0$,

$$V(s, y, a) = \inf_{u(\cdot)\in\mathcal{U}_a[s,T]} \mathbb{E}\Big[\int_s^{\bar{s}} f(r, x^{u,s,y,a}(r), u(r))dr + V(\bar{s}, x^{u,s,y,a}(\bar{s}), R^a_{\bar{s}-s})\Big],$$

where

$$R^a_t := S_{L_{(t-a)^+}} + a - t, \quad t \ge 0.$$

Hölder regularity for the value function

Theorem (Zhang-C. 2022+)

Under Lipschitz assumption on the coefficients with Lipschitz coefficient L, there is a constant $C = C(T, L, \kappa, \nu) > 0$ *so that for any* $s, \overline{s} \in [0, T)$, $y, \overline{y} \in \mathbb{R}^d$ *and* $a, \overline{a} \in [0, \infty)$

$$\begin{aligned} & |V(s,y,a) - V(\overline{s},\overline{y},\overline{a})| \\ \leq \; & C\Big(|y-\overline{y}| + (1+|y|+|\overline{y}|)|s-\overline{s}| \\ & + (|y|+|\overline{y}|)(|a-\overline{a}| \wedge T) + \sqrt{|s-\overline{s}|} + \sqrt{|a-\overline{a}| \wedge T}\Big). \end{aligned}$$

Hamilton-Jacobi-Bellman equation

Theorem (Zhang-C. 2022+)

Suppose the value function $V(t,x,a)$ is $C^{1,2,1}$-smooth. Then

$$\begin{cases} V_s(s,y,a) - V_a(t,y,a) + \inf_u \left(b(t,y,u)V_y(t,y,a) - f(s,y,u)\right) \\ \quad = 0 \quad \text{for } a > 0, \qquad \textit{(interior eqn.)} \\ V_s(s,y,0) + \frac{1}{\kappa} D_a^\nu V(s,y,0) + \inf_u \Big(b(t,x,u)V_y(t,y,0) \\ \quad + \frac{1}{2\kappa}\sigma^2(s,x,u)V_{yy}(s,y,0) - f(s,y,u) \Big) = 0, \\ \qquad \textit{(when } a = 0\textit{: boundary eqn.)} \\ V(T,x,a) = h(x). \end{cases}$$

where

$$D^\nu v(x) := \int_{(0,\infty)} (v(x+z) - v(x))\nu(dz).$$

Martingale representation theorem

The following result holds for any subordinator S.

Theorem (Zhang-C. 2022+)

For each $a \geq 0$, $T \in (0, \infty]$ and $\xi \in L^2(\mathcal{F}_T^{\prime,a})$, there exists an $\{\mathcal{F}_t^\prime\}_{t\in[0,T]}$-predictable process H_s with $\mathbb{E} \int_0^T H_s^2 dL_{(s-a)^+} < \infty$ so that

$$\xi = \mathbb{E}[\xi] + \int_0^T H_s dB_{(L_s - a)^+}. \tag{0.1}$$

Such H is unique in the sense that if H' is another $\{\mathcal{F}_t^{\prime,a}\}_{t\in[0,T]}$-predictable process, then

$$\mathbb{E} \int_0^T |H_s - H_s'|^2 dL_{(s-a)^+} = 0.$$

Theorem (Zhang-C. 2022+)

Under the Lipschitz conditions, for any $a \geq 0$, $T > 0$ and and $\xi \in L^2(\mathcal{F}_T^{l,a})$, the BSDE

$$dY_t = h_1(t, Y_t)dt + h_2(t, Y_t, Z_t)dL_{(t-a)^+} + Z_t dB_{L_{(t-a)^+}}$$

with $Y_T = \xi$ admits a unique adapted square-integrable solution (Y, Z).

We can then use it to study stochastic maximum principle using spiking variational method.

BSDEs driven by sub-diffusions

Theorem (Zhang-C. 2022+)

Under the Lipschitz conditions, for any $a \geq 0$, $T > 0$ and and $\xi \in L^2(\mathcal{F}_T^{I,a})$, the BSDE

$$dY_t = h_1(t, Y_t)dt + h_2(t, Y_t, Z_t)dL_{(t-a)^+} + Z_t dB_{L_{(t-a)^+}}$$

with $Y_T = \xi$ admits a unique adapted square-integrable solution (Y, Z).

We can then use it to study stochastic maximum principle using spiking variational method.

Stochastic maximum principle for spiking variations

Theorem (Zhang-C. 2022+)

Let $(\overline{u}(\cdot), \overline{x}(\cdot))$ be an optimal pair for the control problem with $s = 0$ in $\mathcal{U}'_a[0, T]$. Let (p, q) and (P, Q) be the solutions to the company BSDEs. Then for every $v \in \mathcal{U}'_a[0, T]$ and $t \in (0, T]$,

$$\mathbb{E}\left[\delta b(t)p(t) - \delta f(t)\right] + \kappa^{-1}\mathbb{E}\left[\mathbb{1}_{\{R^a(t)=0\}}\left(\delta\sigma(t)q(t) + \frac{1}{2}(\delta\sigma(t))^2 P(t)\right)\right] \leq 0.$$

Here $\delta\varphi(t) := \varphi(t, \overline{x}(t), v(t)) - \varphi(t, \overline{x}(t), \overline{u}(t))$.

$$\begin{cases} dp(t) &= -\left(b_x(t, \overline{x}(t), \overline{u}(t))p(t) - f_x(t, \overline{x}(t), \overline{u}(t))\right) dt - \sigma_x(t, \overline{x}(t), \overline{u}(t))q(t)dL_{(t-a)^+} \\ & \quad + q(t)dB_{L_{(t-a)^+}} \qquad \text{for } t \in [0, T], \\ p(T) &= -h'(x(T)). \end{cases}$$

$$\begin{cases} dP(t) &= -\left(2b_x(t, \overline{x}(t), \overline{u}(t))P(t) + b_{xx}(t, \overline{x}(t), \overline{u}(t))p(t) - f_{xx}(t, \overline{x}(t), \overline{u}(t))\right) dt \\ & \quad - \left((\sigma_x(t, \overline{x}(t), \overline{u}(t)))^2 P(t) + 2\sigma_x(t, \overline{x}(t), \overline{u}(t))Q(t) + \sigma_{xx}(t, \overline{x}(t), \overline{u}(t))q(t)\right) dL_{(t-a)^+} \\ & \quad + Q(t)dB_{L_{(t-a)^+}} \\ P(T) &= -h''(\overline{x}(T)). \end{cases}$$

Stochastic maximum principle

Recall the cost functional for control $u \in \mathcal{U}'_a[s, T]$ is

$$J(s, x_0, u, a) = \mathbb{E}\left[\int_s^T f\left(t, x^{u,s,x_0,a}(t), u(t)\right) dt + h(x^u(T)) \right],$$

and the optimal control u^* is to minimize this cose, that is,

$$J(s, x_0, u^*, a) = \inf_{u \in \mathcal{U}'_a[s,T]} J(s, x_0, u, a).$$

Theorem (Zhang-C. 2022+)

Under the Lipschitz conditions, suppose that $u^*(\cdot)$ is an optimal and $x^*(\cdot)$ be the corresponding state process

$$x^*(t) = \; b(t, x^*(t), u^*(t))dt + \sigma(t, x^*(t), u^*(t))dB_{L_{(t-s-a)^+}},$$

then for every $t \in [0, T)$, almost surely

$$b_u(t, x^*(t), u^*(t))p(t) + \kappa^{-1}\sigma_u(t, x^*(t), u^*(t))q(t)\mathbb{1}_{\{R^a(t)=0\}} \\ -f_u(t, x^*(t), u^*(t)) = 0.$$

Thank you!